Tutorial 2 (Abubakari Sumaila Salpawuni)

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Q_1 – solution

Given that $X_i \stackrel{iid}{\sim} Unif(\theta, \theta + |\theta|), \quad \theta \neq 0$

For moments, generally, $M_k^* = \frac{1}{n}\sum_{i=1}^n X_i^k$ is the k^{th} sample moment, for $k = 1, 2, \ldots$.
This implies $M_1^* = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}$

$$\begin{aligned}
E(X) &= M_1^* \\
&= \frac{\theta + \theta + |\theta|}{2} = M_1^* \\
&= \theta + \frac{|\theta|}{2} = M_1^* \\
&= \frac{3}{2}\theta I_{(\theta>0)} + \frac{\theta}{2} I_{(\theta<0)}
\end{aligned}$$

Note that $M_1^* = \bar{X}$, and $P(\bar{X} > 0|\theta > 0) = P(\bar{X} < 0|\theta < 0) = 1$. Solving for each condition (on $\theta \neq 0$), we have;

$$\therefore \hat{\theta} = \frac{2}{3}\bar{X} I_{(\bar{X}>0)} + 2\bar{X} I_{(\bar{X}<0)}$$

b) The distribution of X_i can be considered for both sides (i.e, $\theta > 0$ and $\theta < 0$ such that;

$$X_i \sim \begin{cases} Uniform(\theta, 2\theta) & \text{if} \quad \theta > 0 \\ \\ Uniform(\theta, 0) & \text{if} \quad \theta < 0 \end{cases}$$

Their likelihood functions are thus,

$$f_X(x;\theta) = \begin{cases} \frac{1}{\theta^n} I\left(\frac{X_{(n)}}{2} \leq \theta \leq X_{(1)}\right) & \text{if} \quad \theta > 0 \\ \\ \frac{1}{|\theta^n|} I\left(\theta \leq X_{(1)}\right) & \text{if} \quad \theta < 0 \end{cases}$$

The maximum likelihood estimator, θ, is therefore the *order statistic*;

$$\begin{cases} \underset{\theta>0}{argmax} \quad f_X(x;\theta) = \frac{X_{(n)}}{2} \\ \\ \underset{\theta>0}{argmax} \quad f_X(x;\theta) = X_{(1)} \end{cases}$$

$$\therefore \hat{\theta}_{\text{MLE}} = \frac{X_{(n)}}{2} I_{(X_1>0)} + X_{(n)} I_{(X_1<0)}$$

Q_2 – solution

Check to see if there exists a mode, equate $f'(x)$ to zero (maximum value), checking that $f''(x) < 0$

$$\begin{aligned} f(x) = \frac{4}{81}x(9-x^2) &\implies f'(x) = \frac{\partial f(x)}{\partial x} = \frac{1}{81}(36-12x^2) \\ &\implies f''(x) = -\frac{24x}{81} < 0 \quad \text{(mode extists)} \end{aligned}$$

a) at the mode;

$$\begin{aligned} &f(x) = 0 \\ &\frac{4}{81}(9-3x^2) = 0 \\ &\therefore x = \sqrt{3} \end{aligned}$$

b) median

$$\begin{aligned} F(m) &= 0.5 \\ P(x \leq m) &= 0.5 \\ &= \frac{4}{81}\int_0^m (9x - x^3) = 0.5 \\ &= \frac{4}{81}\left(\frac{9}{2}x^2 - \frac{1}{4}x^4\right)\Bigg|_0^m = 0.5 \\ &= 72m^2 - 4m^4 = 162 \\ &= 2m^4 - 36m^2 + 81 = 0 \end{aligned}$$

Solving quadratically in m^2, i.e., ($ax^2 + bx + c = 0$);

$$\begin{aligned} m^2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{36 \pm \sqrt{36^2 - 4 \cdot 2 \cdot 81}}{2 \cdot 81} \\ &= 9 - \frac{9}{2}\sqrt{2} \quad \text{or} \quad 9 + \frac{9}{2}\sqrt{2} \\ m^2 &= 9 - \frac{9}{2}\sqrt{2} \\ m &= \sqrt{9 - 9/2\sqrt{2}} = 1.6236 \\ \therefore \quad &\text{the median is} \quad \mathbf{1.6236} \end{aligned}$$

c) compare

$$E(X) = \int f(x)dx = \int_0^3 x \cdot \frac{4}{81} x(9 - x^2)dx$$
$$= \frac{4}{81}\left(3x^3 + \frac{1}{5}x^5\right)\Big|_0^3$$
$$= \mathbf{1.60}$$

Since mean < median < mode, the distribution of X is said to be *skewed* to the left.

Q_3 – solution

Since $\bar{X}$ and S^2 are independent, we have;

$$P(0 < \bar{X} < 6, 55.22 < S^2 < 145.6) = P(0 < \bar{X} < 6) \times P(55.22 < S^2 < 145.6)$$

$$P(0 < \bar{X} < 6) = P\left(\frac{0-3}{\sqrt{\frac{100}{25}}} < z < \frac{6-3}{\sqrt{\frac{100}{25}}}\right)$$
$$= P(-1.5 < z < 1.5)$$
$$= 0.8664$$

$$P(55.22 < S^2 < 145.6) = P\left(\frac{155.2 \times 25}{100} < \frac{nS^2}{\sigma^2} < \frac{145.6 \times 25}{100}\right)$$
$$= P(13.8 < \chi^2_{24} < 36.8)$$
$$= 0.95 - 0.05$$
$$= 0.90$$

$$\therefore P(0 < \bar{X} < 6, 55.22 < S^2 < 145.6) = 0.8664 * 0.90 = \mathbf{0.8231}$$